Binomial Law of Large Numbers

Ex: Suppose you toss a fair coin 100 times. What is the probability that you will see tails more than 51% of the time.

$$S_{100} \sim \text{Binomial}(\overset{\text{trials}}{100}, \overset{\text{fairness}}{\tfrac{1}{2}})$$

no upper bound

$$P(\underset{\#\text{ of tails}}{S_{100}} > (0.51)(100))$$

approximate

mean $n\,p$

$$\mu = 100 \cdot \frac{1}{2} = 50$$

variance $n\,p\,(1-p)$

$$\sigma^2 = 100 \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$\Rightarrow \sigma = 10 \cdot \frac{1}{2} = 5$$

Recall the form of the CLT: (Central Limit Theorem)

standardization

$$P\left(\frac{a-\mu}{\sigma} \leq \frac{S_{100}-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}\right) \approx \int_a^b \phi \, dx$$

$a \to 51$, $\mu \to 50$, $\sigma \to 5$, $b \to \infty$

pdf of a Normal random var.

$$\mu = np = 50$$

$$\sigma = \sqrt{np(1-p)} = 5$$

integrating the pdf $\Rightarrow$ cdf of a $N(0,1)$ random variable.

$$P\left(\frac{S_{100}-50}{5} > \frac{51-50}{5}\right) \approx \Phi(\infty) - \Phi\left(\frac{1}{5}\right)$$

$$= 1 - \Phi\left(\frac{1}{5}\right)$$

$$\frac{1}{5} = 0.2$$

$$= 1 - 0.5793$$

$$P\left(\frac{0.51\cdot\overset{n}{\widetilde{100}} - \overset{n(0.5)}{\mu}}{\sigma} \le \frac{S_{100} - \mu}{\sigma} \le \infty\right)$$

$$\approx \int^{\infty} \phi(x)\,dx = \Phi(\infty) - \Phi(1/5) =$$

$$\frac{0.01\cdot\cancel{100}}{\sqrt{\cancel{100}_{n}\cdot 1/2\cdot 1/2}}$$

$$= 1 - \Phi\left(\frac{n(0.51) - \overbrace{n\cdot(0.5)}^{\mu}}{\sqrt{np(1-p)}}\right)$$

Let us plot this on a computer (Like a Jupyter notebook)

$$= 1 - \Phi\left(\frac{\cancel{n}^{\sqrt{n}}\cdot \overset{(0.51-0.50)}{(0.01)}}{\sqrt{\cancel{n}\cdot 1/2\cdot 1/2}}\right) = 1 - \Phi\left(\frac{\sqrt{n}\cdot(0.01)}{1/2}\right) \quad n = 100$$

$$= 1 - \Phi\left(2\sqrt{n}\,0.01\right)$$

I made n a variable and plotted it.

$$\underbrace{P(S_n > 0.51n) \quad \text{as} \quad n \to \infty}_{\longrightarrow 0} \quad \text{(LOTS of trials)}$$

As n goes to ∞, the probability of seeing more than 51% tails in your coin tosses goes to 0.

Deviations from the expected value

As n goes to ∞, the probability of seeing less than 49% tails in your coin tosses goes to 0.

What have we seen so far?

Let $S_n \sim \text{Binomial}(n, 1/2)$

$$P(S_n > 0.51n) \longrightarrow 0 \qquad (n \to \infty)$$

Similarly

$$P(S_n < \overset{49\%}{\overbrace{0.49}}n) \longrightarrow 0 \qquad (n \to \infty)$$

saw on your midterms

Recall $\overbrace{P(A \cup B) \leq P(A) + P(B)}$

$$\Rightarrow \quad P(S_n - 0.5n > 0.01n \;\cup\; S_n - 0.5n < -0.01n)$$

0.50n

Expected value

$$\leq P(S_n - 0.5n > 0.01n) + P(S_n - 0.5n < -0.01n)$$

$$\underbrace{P(S_n > 0.51n)}_{} \quad + \quad P(S_n < 0.49n)$$

$$\longrightarrow 0$$

$$\Rightarrow = P(|S_n - 0.5n| > 0.01n) \longrightarrow 0$$

deviates from its expected value

Theorem: LAW OF LARGE NUMBERS

Let $S_n \sim \text{Binomial}(n,p)$

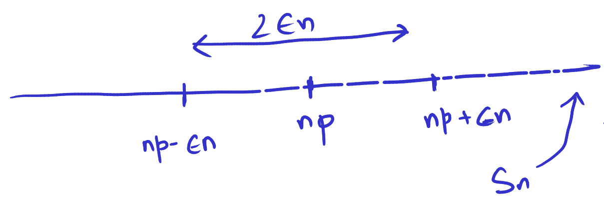
"error or deviation"

$$P\left(|S_n - np| > \epsilon n\right) \longrightarrow 0$$

complements of eachother.

OR equivalently

$$\leftrightarrow P\left(|S_n - np| < \epsilon n\right) \longrightarrow 1$$



(extremely small)

Pf:

$$P(|S_n \underbrace{- np}_{\text{subtract mean}}| < \epsilon n) = P\left(\frac{|S_n - np|}{\underbrace{\sqrt{np(1-p)}}_{\text{divide by } \sqrt{\text{variance}}}} < \frac{\epsilon n}{\sqrt{np(1-p)}}\right)$$

$$\leq \Phi(\infty) - \Phi\left(\frac{\epsilon n}{\sqrt{np(1-p)}}\right) + \Phi\left(\frac{-\epsilon n}{\sqrt{np(1-p)}}\right) - \Phi(-\infty)$$

$$\left(\frac{-\epsilon \cancel{n}\sqrt{n}}{\sqrt{\cancel{n}p(1-p)}} < \frac{S_n - np}{\sqrt{np(1-p)}} < \frac{\epsilon \cancel{n}\sqrt{n}}{\sqrt{\cancel{n}p(1-p)}}\right) = \int_{\frac{-\epsilon\sqrt{n}}{\sqrt{p(1-p)}}}^{\frac{\epsilon\sqrt{n}}{\sqrt{p(1-p)}}} \underbrace{\phi(x)}_{\text{pdf of } N(0,1)}\, dx$$

$$\xrightarrow{n \to \infty} \int_{-\infty}^{\infty} \phi(x)\, dx = 1$$

6/35

attendance.

POLL

Was the proof sketch of the law of large numbers understandable?

A YES

B NO